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# Exercises for 'Functional Analysis 2' [MATH-404]

(07/04/2025)

**Ex 7.1 (Examples of distributions)**

a) Given $\ell > 0$ and $\varphi \in \mathcal{D}(\mathbb{R})$ define the Dirac comb by

$$T(\varphi) = \sum_{n\in\mathbb{Z}} \varphi(n\ell).$$

Show that $T \in \mathcal{D}'(\mathbb{R})$.

b) For $\varphi \in \mathcal{D}(\mathbb{R})$ we set

$$T(\varphi) = \text{p.v.}\left(\frac{1}{x}\right)(\varphi) := \lim_{\varepsilon\to 0}\int_{\mathbb{R}\setminus(-\varepsilon,\varepsilon)} \frac{\varphi(x)}{x}\,\mathrm{d}x.$$

Show that the limit exists and that $T$ is a distribution. This is called the principal value of $1/x$.

c) Define the distribution $T \in \mathcal{D}'(\mathbb{R})$ as $T(\varphi) = \sum_{n\in\mathbb{N}} D^{(n)}\varphi(n)$. Show that $T$ has infinite order.

**Ex 7.2 (Further illustrative examples of distributions)**

a) Let $f \in C^\infty(\mathbb{R})$ be defined by $f(x) = x$. Show that as distributions

$$(\delta_0 \cdot f)\cdot \text{p.v.}\left(\frac{1}{x}\right) = 0 \neq \delta_0 \cdot (f \cdot \text{p.v.}\left(\frac{1}{x}\right)) = \delta_0.$$

Here $\delta_0$ denotes the Dirac delta distribution defined by $\delta_0(\varphi) = \varphi(0)$.

**Remark**: In the lecture we wrote the product of a distribution and a smooth function as $\psi T$. However, this is just a matter of notation and terms of the form $T\psi$ should be interpreted the same way. Moreover, in this exercise you should identify smooth functions with their associated distribution. It shows that we cannot define an associative product on distributions that extends the product of smooth functions and distributions.

b) Let $E \subset \mathbb{R}^d$ be a bounded, open set with $C^1$-boundary. Then its characteristic function $\chi_E$ belongs to $L^1(\mathbb{R}^d)$. Show that its distributional gradient $\nabla\chi_E$ in $\mathcal{D}'(\mathbb{R}^d)$ is given by

$$\nabla\chi_E = -\nu_{\partial E}(x)\mathcal{H}^{d-1}|_{\partial E},$$

where $\nu_{\partial E}$ denotes the exterior normal vector to $\partial E$ and $\mathcal{H}^{d-1}|_{\partial E}$ is the surface measure on $\partial E$.

**Hint**: Recall the divergence theorem.

c) Consider a sequence $\chi_j \in \mathcal{D}'(\mathbb{R})$ of functions defined by $\chi_j(x) = j$ if $|x| < (2j)^{-1}$ and $\chi_j(x) = 0$ otherwise. Show that $\chi_j \to \delta_0$ in $\mathcal{D}'(\mathbb{R})$ but $\chi_j^2$ does not converge in $\mathcal{D}'(\mathbb{R})$ as $j \to \infty$. Why does this example indicate that we cannot make sense of the square of $\delta_0$ in $\mathcal{D}'(\mathbb{R})$ ?

**Ex 7.3 (Distributions with vanishing derivatives - the 1D case)**

Let $I = (a, b) \subset \mathbb{R}$ be an open interval.

a) Let $\varphi \in \mathcal{D}(I)$. Show that there exists $\psi \in \mathcal{D}(I)$ with $\psi' = \varphi$ if and only if $\int_I \varphi \, \mathrm{d}y = 0$.
b) Conclude that for every distribution $T \in \mathcal{D}'(I)$ with $T' = 0$ there exists a constant $c \in \mathbb{R}$ such that $T = c$ in $\mathcal{D}'(I)$.

**Remark**: The result in 7.3 b) is still valid on connected open sets in higher dimensions when the distributional gradient vanishes. However, the proof is more technical.

**Ex 7.4 (PDEs and the fundamental lemma of the calculus of variations*)**
In the course we will show that for a function $y \in L^1_{\text{loc}}(\Omega)$ the property

$$\int_\Omega y(x)\varphi(x) \, \mathrm{d}x = 0 \quad \forall \varphi \in \mathcal{D}(\Omega)$$

implies that $y = 0$ almost everywhere. In this exercise we will apply a weaker version to sketch how to find solutions to some PDEs. Let $\Omega \subset \mathbb{R}^d$ be open, bounded with $C^1$-boundary and $g : \partial\Omega \to \mathbb{R}$, $f : \Omega \to \mathbb{R}$ be continuous, bounded functions. Assume that $u : \overline{\Omega} \to \mathbb{R}$ is a solution of the following problem :[1]

$$\min \left\{ F(u) := \int_\Omega \frac{1}{2} |\nabla u(x)|^2 + f(x)u(x) \, \mathrm{d}x : \quad u \in C^1(\overline{\Omega}),\, u = g \text{ on } \partial\Omega \right\}.$$

a) Show that in a distributional sense it holds that $\Delta u = f$ in $\mathcal{D}'(\Omega)$.
b) Show that $u \in C^2(\Omega)$ implies that $\Delta u = f$ pointwise.

**Hint**: Consider for $\varepsilon > 0$ the perturbation $u + \varepsilon\varphi$ with $\varphi \in \mathcal{D}(\Omega)$ and compute the derivative of $F(u + \varepsilon\varphi)$ in $\varepsilon = 0$. For b) use the fundamental lemma.

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1. The existence of such a solution is not trivial and requires ideas from the calculus of variations and regularity theory for PDEs. In general, to have $C^1$-regularity up to the boundary, one needs to assume that $\Omega$ has a regular boundary.